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An approach to generate superextensions of integrable systems

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Abstract. An algorithm to generate integrable systems is extended to the super case. Some new examples of superextensions of integrable systems are illustrated. We also generalize the trace identity due to Tu to the super case and use it to establish Hamiltonian structures of superextensions of integrable systems under consideration.

1. Introduction

The superintegrable systems in general and the superextensions of the standard integrable systems have been recently investigated. Many papers have been dedicated to the subject (see, e.g. [1–8]). For the famous KdV equation, two kinds of extensions are well known: one is the so-called supersymmetric KdV equation derived by Manin and Radul [2]; the other is the Kupershmidt's version [3]. On the other hand, there has been active research on searching for integrable systems based on Lie algebras and Kac–Moody algebras and different methods have been constructed (see, e.g., [9–15]). Furthermore, some of the results are extended to include Lie superalgebra (see, e.g., [4–7]). In [16], we have developed Tu's approach, and an effective algorithm to generate integrable systems is given. In this paper, we generalize the results of [16] to the superextension case. Besides, the so-called Trace identity to Hamiltonian structures of integrable systems [17–20] is also extended to the super case.

This paper is arranged as follows. In the next section, we first introduce some notations and conventions. A simple scheme to generate superextensions of integrable systems is described and an illustrative example is given in detail. Some other interesting examples are considered in section 3. In section 4, we give some examples to show that a hierarchy of equations connected with (1) could also be derived in some special cases although $e_0(\lambda)$ is not so-called pseudoregular. In section 5, the trace identity is generalized to the superextension case and used to establish Hamiltonian structures of superextensions of integrable systems under consideration. Finally, conclusion and remarks are given in section 6.

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2. A scheme to generate superextensions of integrable systems

Let us begin with the Lie superalgebra $sl(m/n)$ which is defined as

$$sl(m/n) = \left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathrm{Str}\, X = \mathrm{Tr}\, A - \mathrm{Tr}\, D = 0 \right\}$$

where A is a $(m \times m)$ matrix, B a $(m \times n)$ matrix, C a $(n \times m)$ matrix, and D a $(n \times n)$ matrix. The super Lie bracket is defined as [21, 22]

$$[X, Y] = XY - (-1)^{P(X)P(Y)} YX \qquad \forall X, Y \in sl(m/n)$$

where parity $P(X)$ of X is defined as

$$P(X) = \begin{cases} 0 & X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \quad \mathrm{Str}\, X = 0 \\ 1 & X = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \end{cases}$$

Here $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ are even (bosonic) elements and $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ are odd (fermionic) ones. For $sl(m/n)$, the corresponding superloopalgebra is

$$G = sl(m/n) \otimes C[\lambda, \lambda^{-1}] \qquad \lambda \text{ is an even parameter.}$$

The Lie super bracket of G is defined as

$$[X \otimes \lambda^m, Y \otimes \lambda^n] \equiv [X, Y] \otimes \lambda^{m+n}.$$

Different gradations of G may be available. In what follows, we always fix the gradation as a natural gradation, i.e.

$$\deg(X \otimes \lambda^n) = n \qquad \forall X \in sl(m/n).$$

We now consider the following spectral problem

$$\psi_x = U\psi \tag{1}$$

where

$$\psi = (\psi_1, \ldots, \psi_m, \psi_{m+1}, \ldots, \psi_{m+n})^{\mathrm{T}}$$

ψ_i $(i = 1, \ldots, m)$ is an even variable, ψ_i $(i = m+1, \ldots, m+n)$ is an odd variable, and $U = e_0(\lambda) + u_1 e_1(\lambda) + \cdots + u_p e_p(\lambda)$. Here $e_i(\lambda)(i = 0, 1, \ldots, p) \in G$ and $e_i(\lambda)(i = 1, \ldots, p)$ is an even or odd element. u_i is an even (odd) variable if $e_i(\lambda)$ is an even (odd) element. Similar to [14–18], we assume that $e_i(\lambda)(i = 0, \ldots, p)$ meets the conditions:

(i) $e_0(\lambda), e_1(\lambda), \ldots, e_p(\lambda)$ are linearly independent;

(ii) $e_0(\lambda)$ is even and pseudoregular, i.e.

$$G = \mathrm{Kerad}\ e_0(\lambda) \oplus \mathrm{Imad} e_0(\lambda)$$

Kerad $e_0(\lambda)$ is commutative where

$$\mathrm{Kerad}\ e_0(\lambda) = \{X | X \in G, [X, e_0(\lambda)] = 0\}$$

$$\mathrm{Imad}\ e_0(\lambda) = \{Y \in G, \text{s.t. } Y = [X, e_0(\lambda)]\}$$

(iii) $d_0 > 0, d_0 > d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$, where $d_i = \deg e_i(\lambda)$. A simple scheme for generating superextensions of integrable systems can easily be copied from [16–19]. The scheme contains two steps. First, we take a solution $V = \sum_{i=0}^{\infty} V_i \lambda^{-i}$ of the co-adjoint equation associated with (1),

$$V_x = [U, V]. \tag{2}$$

Second, we search for a $\Delta_n \in G$ such that for

$$V^{(n)} \equiv (\lambda^n V)_+ + \Delta_n$$

the following holds:

$$-V_x^{(n)} + [U, V^{(n)}] \in Ce_1(\lambda) + Ce_2(\lambda) + \cdots + Ce_p(\lambda).$$

This requirement yields a hierarchy of superextensions of evolution equations:

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0.$$

Here and in the following, we always denote $(\lambda^n V)_+ \equiv \sum_{i=0}^{n} V_i \lambda^{n-i}$. Similar to the proof of proposition 2 in [16] and replacing the second equation of (3′) in [16] with

$$(\mathrm{Str}\, V^k)_x = 0, \;\; k = 1, \ldots, m+n$$

we can obtain the following result.

Proposition 1. There exists a non-zero $V = \sum_{i=0}^{\infty} V_i \lambda^{-i} \in G$ such that (2) holds and elements of matrix V_i are all pure polynomials of $u_i (i = 1, \ldots, p)$ and their derivatives.

From proposition 1, we deduce that

$$-(\lambda^n V)_{+x} + [U, (\lambda^n V)_+] = (\lambda^n V)_{-x} - [U, (\lambda^n V)_-] \tag{3}$$

where $(\lambda^n V)_+ \equiv \sum_{i=0}^{n} V_i \lambda^{n-i}$ and $(\lambda^n V)_- = \lambda^n V - (\lambda^n V)_+$. It is easy to see that the terms on the left-hand side of (3) are of degrees not less than $d_p^- \equiv (d_p - |d_p|)/2$, while the terms on the right-hand side are of degrees not greater than $d_0 - 1$, therefore the terms on both sides are of degrees ranging over the interval $\delta = [d_p^-, d_0 - 1]$. Thus we deduce that

$$-(\lambda^n V)_{+x} + [U, (\lambda^n V)_+] = \sum_{i \in \delta} f_i$$

for some $f_i \in G_i \equiv \{x \,|\, \deg x = i,\;\; x \in G\}$. Therefore, when we take $e_1(\lambda), \ldots, e_p(\lambda)$ as a basis of $\oplus_{i \in \delta} G_i$, we could in general derive a hierarchy of integrable equations. In order to reduce the number of potentials, we need to consider the reduced spectral problems of (1) and various subalgebras of G. In the following, we give an illustrative example to describe the scheme.

Example 1. Consider the subalgebra $b(0, 1)$ of $sl(2/1)$. Its basis is

$$E_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

where E_0, E_1, E_2 are even elements and E_3, E_4 are odd ones. Their non-zero (anti)commutation relations are

$$[E_0, E_1] = 2E_1 \qquad [E_0, E_2] = -2E_2 \qquad [E_0, E_3] = E_3 \qquad [E_0, E_4] = -E_4$$
$$[E_1, E_2] = E_0, \qquad [E_1, E_4] = E_3 \qquad [E_2, E_3] = E_4 \qquad [E_3, E_3] = -2E_1$$
$$[E_3, E_4] = E_0 \qquad [E_4, E_4] = 2E_2.$$

Set G_1 as a linear span of $\{E_0 \otimes \lambda^{dn},\ E_1 \otimes \lambda^{dn+e_1},\ E_2 \otimes \lambda^{dn+e_2},\ E_3 \otimes \lambda^{dn+e_3},\ E_4 \otimes \lambda^{dn+e_4} | n \in Z, d$ is an integer greater than 1 and e_i are integers such that $0 \leqslant e_i < d\ (i = 1, 2, 3)\}$. We can easily verify that G_1 is a subalgebra of $b(0,1) \otimes C[\lambda, \lambda^{-1}]$ iff

$$e_1 + e_2 = 0 \pmod d \qquad e_1 + e_4 = e_3 \pmod d \qquad e_3 + e_4 = 0 \pmod d$$
$$2e_3 = e_1 \pmod d \qquad 2e_4 = e_2 \pmod d. \tag{4}$$

Here we only consider two cases.

Case I. We have the following solution of (4): $d = 2, e_1 = e_2 = 0, e_3 = e_4 = 1$. Furthermore, taking $d_0 = 2, d_p = 0$, and $e_0(\lambda) = E_0 \otimes \lambda^2$, we get the following new spectral problem

$$\psi_x = U\psi$$

where

$$U = E_0\lambda^2 + uE_1 + vE_2 + wE_0 + \epsilon E_3\lambda + \beta E_4\lambda = \begin{bmatrix} \lambda^2 + w & u & \lambda\epsilon \\ v & -\lambda^2 - w & \lambda\beta \\ \lambda\beta & -\lambda\epsilon & 0 \end{bmatrix}.$$

Here u, v, and w are even potentials and ϵ and β are odd ones. Set

$$V = aE_0 + bE_1 + cE_2 + dE_3 + eE_4 = \begin{bmatrix} a & b & d \\ c & -a & e \\ e & -d & 0 \end{bmatrix}$$
$$= \sum_{n \geqslant 0} (a_n\lambda^{-2n}E_0 + b_n\lambda^{-2n}E_1 + c_n\lambda^{-2n}E_2 + d_n\lambda^{-2n-1}E_3 + e_n\lambda^{-2n-1}E_4)$$

where a, b, and c are even and d and e are odd. From $V_x = [U, V]$, we deduce that

$$\begin{aligned} a_x &= uc + \lambda\epsilon e + \lambda\beta d - vb \\ b_x &= -2ua + 2\lambda^2 b - 2\lambda\epsilon d + 2wb \\ c_x &= -2\lambda^2 c + 2va + 2\lambda\beta e - 2wc \\ d_x &= \lambda^2 d + ue - \lambda\epsilon a - \lambda\beta b + wd \\ e_x &= -\lambda^2 e + \lambda\beta a - \lambda\epsilon c + vd - we \end{aligned} \tag{5}$$

or

$$\begin{aligned} a_{mx} &= uc_m + \epsilon e_m + \beta d_m - vb_m \\ b_{mx} &= -2ua_m + 2b_{m+1} - 2\epsilon d_m + 2wb_m \\ c_{mx} &= -2c_{m+1} + 2va_m + 2\beta e_m - 2wc_m \\ d_{mx} &= d_{m+1} + ue_m - \epsilon a_{m+1} - \beta b_{m+1} + wd_m \\ e_{mx} &= -e_{m+1} + \beta a_{m+1} - \epsilon c_{m+1} + vd_m - we_m. \end{aligned} \tag{6}$$

We now give the first few of a_m, b_m, c_m, d_m, and e_m:

$$b_0 = c_0 = 0 \qquad a_0 = k = \text{constant} \neq 0 \ (k \text{ is even})$$
$$d_0 = k\epsilon \qquad e_0 = k\beta \qquad c_1 = kv \qquad b_1 = ku \qquad a_1 = k\beta\epsilon$$
$$d_1 = k(\epsilon_x - w\epsilon) \qquad e_1 = k(-\beta_x - w\beta) \qquad \ldots.$$

In general, we can obtain recursively from (6) all the a_m, b_m, c_m, d_m, and e_m. On the other hand, we have

$$-(\lambda^{2n}V)_{+x} + [U, (\lambda^{2n}V)_+] = -\begin{pmatrix} a_n & b_n & \lambda d_{n-1} \\ c_n & -a_n & \lambda e_{n-1} \\ \lambda e_{n-1} & -\lambda d_{n-1} & 0 \end{pmatrix}_x$$

$$+\left[\begin{pmatrix} w & u & \lambda\epsilon \\ v & -w & \lambda\beta \\ \lambda\beta & -\lambda\epsilon & 0 \end{pmatrix}, \begin{pmatrix} a_n & b_n & 0 \\ c_n & -a_n & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$+\left[\begin{pmatrix} w & u & 0 \\ v & -w & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \lambda d_{n-1} \\ 0 & 0 & \lambda e_{n-1} \\ \lambda e_{n-1} & -\lambda d_{n-1} & 0 \end{pmatrix}\right].$$

Therefore, we can deduce a hierarchy of equations

$$u_t = b_{nx} + 2ua_n - 2wb_n \qquad v_t = c_{nx} - 2va_n + 2wc_n \qquad w_t = a_{nx} - uc_n + vb_n$$
$$\epsilon_t = d_{n-1x} - ue_{n-1} + \epsilon a_n + \beta b_n - wd_{n-1} \qquad \beta_t = e_{n-1x} - \beta a_n + \epsilon c_n - vd_{n-1} + we_{n-1}. \tag{7}$$

In particular, taking $n = 1$ in (7), we have

$$u_t = k(u_x + 2u\beta\epsilon - 2wu) \qquad v_t = k(v_x - 2v\beta\epsilon + 2wv) \qquad w_t = k(\beta\epsilon)_x$$
$$\epsilon_t = k(\epsilon_x - w\epsilon) \qquad \beta_t = k(\beta_x + w\beta).$$

Case II. We have the following solution of (4): $d = 4, e_1 = e_2 = 2, e_4 = 3, e_3 = 1$. Furthermore, take $d_0 = 2, d_p = 0$ and $e_0(\lambda) = (E_1 + E_2) \otimes \lambda^2$, and consider the following new spectral problem

$$\psi_x = U\psi$$

where

$$U = (E_1 + E_2)\lambda^2 + uE_0 + \epsilon E_3\lambda = \begin{bmatrix} u & \lambda^2 & \lambda\epsilon \\ \lambda^2 & -u & 0 \\ 0 & -\lambda\epsilon & 0 \end{bmatrix}.$$

Here u is an even potential and ϵ is an odd one. Set

$$V = aE_0 + bE_1 + cE_2 + dE_3 + eE_4 = \begin{bmatrix} a & b & d \\ c & -a & e \\ e & -d & 0 \end{bmatrix}$$
$$= \sum_{n\geqslant 0}(a_n\lambda^{-4n}E_0 + b_n\lambda^{-4n+2}E_1 + c_n\lambda^{-4n+2}E_2 + d_n\lambda^{-4n+1}E_3 + e_n\lambda^{-4n+3}E_4)$$

where a, b, and c are even and d and e are odd. Then similar to case I, from $U_t - (\lambda^{4n}V)_{+x} + [U, (\lambda^{4n}V)_+] = 0$, we can deduce a hierarchy of equations

$$u_t = a_{nx} \qquad \epsilon_t = d_{nx} - ud_n + \epsilon a_n \tag{8}$$

where all the a_m, d_m can be calculated recursively from the following relations

$$a_{mx} = c_{m+1} - b_{m+1} + \epsilon e_{m+1} \qquad b_{mx} = -2a_m + 2ub_m - 2\epsilon d_m \qquad c_{mx} = 2a_m - 2uc_m$$
$$d_{mx} = e_{m+1} + ud_m - \epsilon a_m \qquad e_{mx} = d_m - ue_m - \epsilon c_m \tag{9}$$

with the initial values $e_0 = 0$, $b_0 = c_0 = k =$ constant $\neq 0$ (k is even). For example, the first few of a_m, b_m, c_m, d_m, and e_m are

$$a_0 = ku \qquad d_0 = k\epsilon \qquad e_1 = k\epsilon_x \qquad b_1 = \tfrac{1}{2}(-ku^2 - ku_x - k\epsilon\epsilon_x)$$

$$c_1 = \tfrac{1}{2}(-ku^2 + ku_x - 3k\epsilon\epsilon_x) \qquad a_1 = \tfrac{1}{4}k(u_{xx} - 3\epsilon\epsilon_{xx}) - \tfrac{1}{2}ku(u^2 + 3\epsilon\epsilon_x)$$

$$d_1 = k(\epsilon_{xx} + u\epsilon_x + \tfrac{1}{2}\epsilon u_x - \tfrac{1}{2}\epsilon u^2) \qquad e_2 = k(\epsilon_{xxx} + \tfrac{3}{2}u_x\epsilon_x + \tfrac{3}{4}\epsilon u_{xx} - \tfrac{3}{2}u^2\epsilon_x - \tfrac{3}{2}\epsilon uu_x)$$

$$b_2 + c_2 = k[-\tfrac{1}{2}(uu_{xx} - \tfrac{1}{2}u_x^2) + \tfrac{3}{2}u\epsilon\epsilon_{xx} - \tfrac{9}{2}u_x\epsilon\epsilon_x$$
$$+\tfrac{3}{4}u^4 - 2(\epsilon\epsilon_{xxx} - \epsilon_x\epsilon_{xx}) + 6\epsilon\epsilon_x u^2] \qquad \ldots.$$

Thus, taking $n = 1$ in (8), we, in particular, have

$$\begin{aligned} u_t &= k(\tfrac{1}{4}u_{xxx} - \tfrac{3}{4}\epsilon_x\epsilon_{xx} - \tfrac{3}{4}\epsilon\epsilon_{xxx} - \tfrac{3}{2}u^2u_x - \tfrac{3}{2}u_x\epsilon\epsilon_x - \tfrac{3}{2}u\epsilon\epsilon_{xx}) \\ \epsilon_t &= k(\epsilon_{xxx} + \tfrac{3}{2}u_x\epsilon_x + \tfrac{3}{4}\epsilon u_{xx} - \tfrac{3}{2}u^2\epsilon_x - \tfrac{3}{2}\epsilon u u_x). \end{aligned} \tag{10}$$

3. Further examples

In this section, we shall consider some other examples.

Example 2. Consider $sl(2/1)$. Its basis is

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad G_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Their (anti)commutation relations are given in [4]. Set G_2 as a linear span of $\{H_a \otimes \lambda^{dn}, E_b \otimes \lambda^{dn+e_b}, F_b \otimes \lambda^{dn+f_b}, G_b \otimes \lambda^{dn+g_b} | n \in Z; a, b = 1, 2; d$ is a positive integer, e_b, f_b and g_b are fixed integers. Obviously, G_2 is a subsuperalgebra of G iff

$$e_a + f_a = 0 \pmod d,\ a = 1, 2;\ e_1 + e_2 = g_1 \pmod d,\ e_1 + g_2 = f_2 \pmod d. \tag{11}$$

It is easy to give the following two solutions of (11).

Case I. $d = 2, e_1 = e_2 = f_1 = f_2 = 1, g_1 = g_2 = 0$.

Case II. $d = 4, e_1 = f_1 = 2, e_2 = g_2 = 1, f_2 = g_1 = 3$.

Let us first consider case I. In this case, we take

$$e_0(\lambda) = (E_1 + F_1) \otimes \lambda$$

and consider the following spectral problem

$$\psi_x = U\psi$$

where

$$U = (E_1 + F_1)\lambda + uH_1 + \epsilon G_1 + \beta G_2 = \begin{bmatrix} u & \lambda & \epsilon \\ \lambda & -u & 0 \\ \beta & 0 & 0 \end{bmatrix}$$

with u being an even potential and ϵ and β odd ones. Set

$$\begin{aligned} V &= aH_1 + bH_2 + cE_1 + dF_1 + eG_1 + fG_2 + gE_2 + hF_2 = \begin{bmatrix} a & c & e \\ d & -a+b & g \\ f & h & b \end{bmatrix} \\ &= \sum_{n\geqslant 0}(a_n\lambda^{-2n}H_1 + b_n\lambda^{-2n}H_2 + e_n\lambda^{-2n}G_1 + f_n\lambda^{-2n}G_2 + g_n\lambda^{-2n-1}E_2 \\ &\quad + c_n\lambda^{-2n-1}E_1 + d_n\lambda^{-2n-1}F_1 + h_n\lambda^{-2n-1}F_2) \end{aligned}$$

where a, b, c, and d are even and e, f, g, and h are odd. Similar to example 1 in section 2, from $U_t - (\lambda^{2n}V)_{+x} + [U, (\lambda^{2n}V)_+] = 0$ we can deduce the corresponding hierarchy of equations

$$u_t = a_{nx} - \epsilon f_n + e_n\beta \qquad \epsilon_t = e_{nx} - ue_n + (a_n - b_n)\epsilon \qquad \beta_t = f_{nx} + uf_n + (b_n - a_n)\beta \tag{12}$$

where all the a_m, b_m, e_m, f_m can be calculated recursively from the following relations:

$$\begin{aligned}
&a_{mx} = d_{m+1} - c_{m+1} + \epsilon f_m + \beta e_m \qquad b_{mx} = \beta e_m + \epsilon f_m \\
&c_{mx} = 2uc_m + (-2a_m + b_m) + \epsilon h_m \qquad d_{mx} = 2a_m - b_m - 2ud_m + \beta g_m \\
&e_{mx} = g_{m+1} + ue_m + \epsilon(b_m - a_m) \qquad f_{mx} = -h_{m+1} + \beta(a_m - b_m) - uf_m \\
&g_{mx} = e_m - ug_m - \epsilon d_m \qquad h_{mx} = -f_m + uh_m + \beta c_m.
\end{aligned} \tag{13}$$

We now give the first few of a_m, b_m, c_m, d_m, e_m, and f_m in two cases.

Case (a). $g_0 = h_0 = 0$. Set $d_0 = c_0 = k = \text{constant} \neq 0$ (k is even). Then

$$\begin{aligned}
&e_0 = k\epsilon \qquad f_0 = k\beta \qquad b_0 = 0 \qquad a_0 = ku \qquad h_1 = -k\beta_x \qquad g_1 = k\epsilon_x \\
&d_1 = \tfrac{1}{2}(ku_x - ku^2 - k\epsilon\beta) \qquad c_1 = \tfrac{1}{2}(-ku_x - ku^2 - k\epsilon\beta) \\
&e_1 = k(\epsilon_{xx} + u\epsilon_x + \tfrac{1}{2}\epsilon u_x - \tfrac{1}{2}\epsilon u^2) \qquad f_1 = k(\beta_{xx} - u\beta_x - \tfrac{1}{2}\beta u_x - \tfrac{1}{2}\beta u^2) \\
&b_1 = k(\epsilon\beta_x - \epsilon_x\beta - u\epsilon\beta) \qquad a_1 = k(\tfrac{1}{4}u_{xx} - \tfrac{1}{4}\epsilon_x\beta + \tfrac{1}{4}\epsilon\beta_x - \tfrac{1}{2}u^3 - u\epsilon\beta) \qquad \ldots.
\end{aligned}$$

Case (b). $g_0 = h_0 = 0$. Set $d_0 = c_0 = 0, a_0 = k = \text{constant} \neq 0$ (k is even). Then

$$\begin{aligned}
&e_0 = f_0 = 0 \qquad g_1 = -k\epsilon \qquad h_1 = -k\beta \qquad d_1 = c_1 = 0 \qquad e_1 = -k\epsilon_x - ku\epsilon \\
&f_1 = k\beta_x - ku\beta \qquad b_1 = k\epsilon\beta \qquad a_1 = 0 \qquad \ldots.
\end{aligned}$$

Corresponding to two different choices of $a_0, b_0, c_0, d_0, e_0, f_0, g_0$, and h_0 we have two hierarchies of equations. In particular, for case (a), taking $n = 0$ in (12), we have

$$\begin{aligned}
u_t &= k(\tfrac{1}{4}u_{xxx} + \tfrac{3}{4}\epsilon_{xx}\beta - \tfrac{3}{4}\epsilon\beta_{xx} - \tfrac{3}{2}u^2u_x) \\
\epsilon_t &= k(\epsilon_{xxx} + \tfrac{3}{2}u_x\epsilon_x + \tfrac{3}{4}u_{xx}\epsilon - \tfrac{3}{2}u^2\epsilon_x - \tfrac{3}{2}uu_x\epsilon + \tfrac{3}{4}\epsilon\epsilon_x\beta) \\
\beta_t &= k(\beta_{xxx} - \tfrac{3}{2}u_x\beta_x - \tfrac{3}{4}u_{xx}\beta - \tfrac{3}{2}u^2\beta_x - \tfrac{3}{2}uu_x\beta + \tfrac{3}{4}\beta\epsilon\beta_x).
\end{aligned}$$

For case (b), taking $n = 1$ in (12), we have

$$\begin{aligned}
u_t &= -k(\epsilon\beta)_x \\
\epsilon_t &= k(-\epsilon_{xx} - u_x\epsilon + u^2\epsilon) \\
\beta_t &= k(\beta_{xx} - u_x\beta - u^2\beta).
\end{aligned}$$

Next we consider case II. In this case, we take

$$e_0(\lambda) = (E_1 + F_1) \otimes \lambda^2$$

and consider the following spectral problem

$$\psi_x = U\psi$$

where

$$U = (E_1 + F_1)\lambda^2 + uH_1 + vH_2 + \epsilon E_2\lambda + \beta G_2\lambda = \begin{bmatrix} u & \lambda^2 & 0 \\ \lambda^2 & -u + v & \lambda\epsilon \\ \lambda\beta & 0 & v \end{bmatrix}$$

with u, v being even potentials and ϵ, β odd ones. Set

$$V = aH_1 + bH_2 + cE_1 + dF_1 + eG_1 + fG_2 + gE_2 + hF_2 = \begin{bmatrix} a & c & e \\ d & -a+b & g \\ f & h & b \end{bmatrix}$$
$$= \sum_{n\geqslant 0}(a_n\lambda^{-4n}H_1 + b_n\lambda^{-4n}H_2 + e_n\lambda^{-4n+3}G_1 + f_n\lambda^{-4n+1}G_2 + g_n\lambda^{-4n+1}E_2$$
$$+c_n\lambda^{-4n+2}E_1 + d_n\lambda^{-4n+2}F_1 + h_n\lambda^{-4n+3}F_2)$$

where $a, b, c,$ and d are even and $e, f, g,$ and h are odd. From $U_t - (\lambda^{2n}V)_{+x} + [U, (\lambda^{2n}V)_+] = 0$, we can deduce a hierarchy of equations

$$u_t = a_{nx} \qquad v_t = b_{nx} \qquad \epsilon_t = g_{nx} + ug_n - \epsilon a_n$$
$$\beta_t = f_{nx} + (u-v)f_n - \beta(a_n - b_n) \tag{14}$$

where all the a_m, b_m, f_m, and g_m can be calculated recursively from the following relations:

$$a_{mx} = d_{m+1} - c_{m+1} + \beta e_{m+1} \qquad b_{mx} = \beta e_{m+1} + \epsilon h_{m+1}$$
$$c_{mx} = -2a_m + b_m + (2u-v)c_m \qquad d_{mx} = 2a_m - b_m + (v-2u)d_m + \epsilon f_m + \beta g_m$$
$$e_{mx} = g_m + (u-v)e_m - \epsilon c_m \qquad f_{mx} = -h_{m+1} + \beta(a_m - b_m) + (v-u)f_m$$
$$g_{mx} = e_{m+1} - ug_m - \epsilon a_m \qquad h_{mx} = -f_m + uh_m + \beta c_m. \tag{15}$$

We now give the first few of $a_m, b_m, c_m, d_m, e_m, f_m, g_m$, and h_m in two cases.

Case (a). $e_0 = h_0 = 0$. Set $d_0 = c_0 = k = \text{constant} \neq 0$ (k is even). Then

$$g_0 = k\epsilon \qquad f_0 = k\beta \qquad b_0 = k\beta\epsilon \qquad a_0 = \tfrac{1}{2}k(2u-v) + \tfrac{1}{2}k\beta\epsilon \qquad \ldots.$$

Case (b). $e_0 = h_0 = 0$. Set $d_0 = c_0 = f_0 = g_0 = 0, a_0 = k = \text{constant} \neq 0$ (k is even), $b_0 = 2k$. Then

$$h_1 = -k\beta \qquad e_1 = -k\epsilon \qquad d_1 = 0 \qquad c_1 = k\epsilon\beta \qquad g_1 = -k\epsilon_x - k(v-u)\epsilon$$
$$f_1 = k\beta_x - ku\beta \qquad b_1 = k(\epsilon_x\beta - \epsilon\beta_x + v\epsilon\beta) \qquad a_1 = -k\epsilon\beta_x + ku\epsilon\beta \qquad \ldots.$$

Corresponding to two different choices of $a_0, b_0, c_0, d_0, e_0, f_0, g_0$ and h_0, we have two hierarchies of equations. In particular, for case (a), taking $n = 0$ in (14), we have

$$u_t = \tfrac{1}{2}k(2u - v + \beta\epsilon)_x \qquad v_t = k(\beta\epsilon)_x \qquad \epsilon_t = k(\epsilon_x + \tfrac{1}{2}v\epsilon) \qquad \beta_t = k(\beta_x - \tfrac{1}{2}v\beta).$$

For case (b), taking $n = 1$ in (14), we have

$$u_t = k(-\epsilon\beta_x + u\epsilon\beta)_x \qquad v_t = k(\epsilon_x\beta - \epsilon\beta_x + v\epsilon\beta)_x$$
$$\epsilon_t = k[-\epsilon_{xx} - (v-u)_x\epsilon - v\epsilon_x - u(v-u)\epsilon] \qquad \beta_t = k[\beta_{xx} - u_x\beta - v\beta_x - u(u-v)\beta].$$

Example 3. $osp(n, 2r)$ is defined as [22]

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \quad \begin{array}{l} A^{\mathrm{T}}G + GA = 0, \\ B^{\mathrm{T}}G - HC = 0, \\ D^{\mathrm{T}}H + HD = 0 \end{array} \quad \begin{array}{c} G = I_n \\ H = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} \end{array} \right\}.$$

It is a subsuperalgebra of $sl(n/2r)$. In the following, we only consider $osp(2, 2)$ for the sake of convenience in the calculation. A basis of $osp(2, 2)$ is as follows,

$$E_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_3=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{bmatrix}\qquad E_4=\begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}\qquad E_5=\begin{bmatrix}0&0&0&1\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{bmatrix}$$

$$E_6=\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&1&0&0\end{bmatrix}\qquad E_7=\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&-1&0&0\\0&0&0&0\end{bmatrix}$$

where E_0, E_1, E_2, and E_3 are even elements and E_4, E_5, E_6, and E_7 are odd ones. Their non-zero (anti)commutation relations are

$$\begin{array}{llll}[E_0,E_4]=-E_6 & [E_0,E_5]=-E_7 & [E_0,E_6]=E_4 & [E_0,E_7]=E_5\\ [E_1,E_2]=2E_2 & [E_1,E_3]=-2E_3 & [E_1,E_4]=-E_4 & [E_1,E_5]=E_5\\ [E_1,E_6]=-E_6 & [E_1,E_7]=E_7 & [E_2,E_3]=E_1 & [E_2,E_4]=-E_5\\ [E_2,E_6]=-E_7 & [E_3,E_5]=-E_4 & [E_3,E_7]=-E_6 & [E_4,E_4]=2E_3\\ [E_4,E_7]=-E_0 & [E_5,E_5]=-2E_2 & [E_5,E_6]=E_0 & [E_6,E_6]=2E_3\\ [E_6,E_7]=-E_1 & [E_7,E_7]=-2E_2. & & \end{array}$$

Set G_4 as a linear span of $\{E_0\otimes\lambda^{dn+e_0},\ E_1\otimes\lambda^{dn},\ E_2\otimes\lambda^{dn+e_2},\ E_3\otimes\lambda^{dn+e_3},\ldots,E_7\otimes\lambda^{dn+e_7}|\ n\in Z, d$ is a positive integer, $e_0,e_2,\ldots,e_7$ are fixed non-negative integers. It is easy to show that G_4 is a subsuperalgebra of $osp(2,2))\otimes C[\lambda,\lambda^{-1}]$ iff

$$\begin{array}{lll}e_0+e_4=e_6\ (\mathrm{mod}\,d) & e_0+e_5=e_7\ (\mathrm{mod}\,d) & e_0+e_6=e_4\ (\mathrm{mod}\,d)\\ e_0+e_7=e_5\ (\mathrm{mod}\,d) & e_2+e_3=0\ (\mathrm{mod}\,d) & e_2+e_4=e_5\ (\mathrm{mod}\,d)\\ 2e_4=e_3\ (\mathrm{mod}\,d) & 2e_5=e_2\ (\mathrm{mod}\,d) & e_5+e_6=e_0\ (\mathrm{mod}\,d).\end{array}\qquad(16)$$

Here we only consider the following solution of (16):

$$d=4\qquad e_0=e_2=e_3=2\qquad e_5=e_6=1\qquad e_4=e_7=3.$$

In this case, a linear span of $\{E_0\otimes\lambda^{4n+2},\ E_1\otimes\lambda^{4n},\ E_2\otimes\lambda^{4n+2},\ E_3\otimes\lambda^{4n+2},\ E_4\otimes\lambda^{4n+3},\ E_5\otimes\lambda^{4n+1},\ E_6\otimes\lambda^{4n+1},\ E_7\otimes\lambda^{4n+3}|\ n\in Z\}$ forms a subsuperloopalgebra of $osp(2,2))\otimes C[\lambda,\lambda^{-1}]$. Take $e_0(\lambda)=(E_0+E_2+E_3)\otimes\lambda^2$. We consider the following new spectral problem

$$\psi_x=U\psi$$

where

$$U=(E_0+E_2+E_3)\lambda^2+uE_1+\epsilon E_5\lambda+\beta E_6\lambda=\begin{bmatrix}0&\lambda^2&0&\lambda\epsilon\\-\lambda^2&0&\lambda\beta&0\\-\lambda\epsilon&0&u&\lambda^2\\0&\lambda\beta&\lambda^2&-u\end{bmatrix}$$

with u being an even potential and ϵ and β odd ones. Set

$$V=aE_0+bE_1+cE_2+dE_3+eE_4+fE_5+gE_6+hE_7=\begin{bmatrix}0&a&e&f\\-a&0&g&h\\-f&-h&b&c\\e&g&d&-b\end{bmatrix}$$
$$=\sum_{n\geqslant0}(a_n\lambda^{-4n+2}E_0+b_n\lambda^{-4n}E_1+c_n\lambda^{-4n+2}E_2+d_n\lambda^{-4n+2}E_3$$
$$+e_n\lambda^{-4n+3}E_4+f_n\lambda^{-4n+1}E_5+g_n\lambda^{-4n+1}E_6+h_n\lambda^{-4n+3}E_7)$$

where a, b, c, and d are even and e, f, g, and h are odd. From $U_t - (\lambda^{2n}V)_{+x} + [U, (\lambda^{2n}V)_+] = 0$, we can deduce a hierarchy of equations

$$u_t = b_{nx} \qquad \epsilon_t = f_{nx} + \epsilon b_n - uf_n \qquad \beta_t = g_{nx} + ug_n - \beta b_n \tag{17}$$

where all the b_m, f_m, and g_m can be calculated recursively from the following relations

$$\begin{aligned} &a_{mx} = \epsilon g_m + \beta f_m \qquad b_{mx} = d_{m+1} - c_{m+1} - \epsilon e_{m+1} - \beta h_{m+1} \\ &c_{mx} = -2b_m + 2uc_m - 2\epsilon f_m \qquad d_{mx} = 2b_m - 2ud_m + 2\beta g_m \\ &e_{mx} = g_m - f_m - ue_m - \beta a_m + \epsilon d_m \qquad f_{mx} = h_{m+1} - e_{m+1} + uf_m - \epsilon b_m \\ &g_{mx} = -e_{m+1} - h_{m+1} - ug_m + \beta b_m \qquad h_{mx} = -g_m - f_m + uh_m + \beta c_m + \epsilon a_m \end{aligned} \tag{18}$$

with initial conditions

$$h_0 = e_0 = 0 \qquad d_0 = c_0 = k_1 = \text{constant} \qquad a_0 = k_2 = \text{constant } (k_i \text{ are even})$$

$$f_0 = \tfrac{1}{2}k_1(\epsilon + \beta) + \tfrac{1}{2}k_2(\epsilon - \beta) \qquad g_0 = \tfrac{1}{2}k_1(\beta - \epsilon) + \tfrac{1}{2}k_2(\epsilon + \beta)$$

$$b_0 = k_1 u + \tfrac{1}{2}(k_2 - k_1)\epsilon\beta \qquad \ldots .$$

Set $n = 0$, then (17) becomes

$$\begin{aligned} u_t &= k_1 u_x + \tfrac{1}{2}(k_2 - k_1)(\epsilon\beta)_x \\ \epsilon_t &= \tfrac{1}{2}(k_1 + k_2)\epsilon_x + \tfrac{1}{2}(k_1 - k_2)\beta_x + \tfrac{1}{2}(k_1 - k_2)u(\epsilon - \beta) \\ \beta_t &= \tfrac{1}{2}(k_1 + k_2)\beta_x + \tfrac{1}{2}(k_2 - k_1)\epsilon_x + \tfrac{1}{2}(k_2 - k_1)u(\epsilon + \beta). \end{aligned}$$

4. Other cases: $e_0(\lambda)$ is not pseudoregular

Note that in sections 2 and 3 we only considered the case when $e_0(\lambda)$ appearing in the spectral problem (1) is so-called pseudoregular. Thus a natural problem arises of whether a hierarchy of equations connected with (1) could be derived when $e_0(\lambda)$ is not so-called pseudoregular. Generally speaking, the answer is negative as, in this case, the co-adjoint equation (2) is not guaranteed to have solution in general. However, for some special cases, it is possible to derive a hierarchy of equations connected with (1) when $e_0(\lambda)$ is not pseudoregular. To illustrate this, let us consider the superalgebra $b(n)$ defined as [22]

$$b(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^{\mathrm{T}} \end{pmatrix} \mid \mathrm{Tr}\, A = 0 \qquad B^{\mathrm{T}} = B \qquad C^{\mathrm{T}} = -C \right\}.$$

In the following, we only consider $b(2)$. A basis of $b(2)$ is as follows,

$$E_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where E_0, E_1, and E_2 are even elements and E_3, E_4, E_5, and E_6 are odd ones. Set G_5 as a linear span of $\{E_0\otimes\lambda^{2n},\ E_1\otimes\lambda^{2n+1},\ E_2\otimes\lambda^{2n+1},\ E_4\otimes\lambda^{2n},\ E_5\otimes\lambda^{2n+1},\ E_6\otimes\lambda^{2n+1},\ n\in Z\}$. It is easy to show that G_5 is a subsuperalgebra of $b(2))\otimes C[\lambda,\lambda^{-1}]$. We now consider the following spectral problem

$$\psi_x = U\psi \tag{19}$$

where

$$U = (E_1+E_2)\lambda + uE_0 + \epsilon E_4 \tag{20}$$

with u being an even potential and ϵ an odd one. It is easily verified that $(E_1+E_2)\lambda$ is not pseudoregular. However, a detailed calculation shows that we can deduce a corresponding hierarchy of equations connected with (19) and (20). Here we only give the first non-trivial equation

$$\begin{aligned} u_t &= \alpha(\tfrac{1}{4}u_{xxx} - u^2u_x) \\ \epsilon_t &= \tfrac{1}{4}\alpha\epsilon_{xxx} - \tfrac{1}{4}ku_{xxx} + \tfrac{3}{2}u^2u_x - \tfrac{3}{2}\alpha u^2\epsilon_x - 3\alpha uu_x\epsilon \end{aligned}$$

where α is an even constant and k is an odd constant.

Similarly, we can deduce corresponding hierarchy of equations connected with the following spectral problem:

$$\psi_x = U\psi = (\lambda E_0 + uE_1 + vE_2 + \epsilon E_5 + \beta E_6)\psi.$$

Here u and v are even potentials and ϵ and β are odd ones. Obviously, λE_0 is not pseudoregular. Besides, we can also consider the superalgebra $d(n)$ defined as [22]

$$d(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} | A \in gl(n),\, B \in sl(n) \right\}.$$

5. Supertrace identity

In this section, we shall present a supertrace identity and use it to establish corresponding Hamiltonian structures of the superextensions of integrable systems under consideration in sections 2 and 3. As the proof of supertrace identity is very similar to that in [17–20], we only give results without proof.

Theorem 2. Suppose that the solution of equation (2) is unique in the sense that two solutions V_1 and V_2 of the same rank differ only by a constant factor: $V_2 = \alpha V_1$, α is an even constant. Then it holds that

$$\frac{\delta}{\delta u_i}\,\mathrm{Str}\left(V\frac{\partial U}{\partial\lambda}\right) = \left(\lambda^{-\gamma}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\gamma}\right)\mathrm{Str}\left(\frac{\partial U}{\partial u_i}V\right) \tag{21}$$

where V satisfies the co-adjoint equation (2).

In what follows, we only give corresponding Hamiltonian structures of (8) as an illustrative application. We can easily obtain that

$$\mathrm{Str}\left(V\frac{\partial U}{\partial\lambda}\right) = \mathrm{Str}\begin{pmatrix} a & b & d \\ c & -a & e \\ e & -d & 0 \end{pmatrix}\begin{pmatrix} 0 & 2\lambda & \epsilon \\ 2\lambda & 0 & 0 \\ 0 & -\epsilon & 0 \end{pmatrix} = 2\lambda b + 2\lambda c - 2e\epsilon$$

$$\mathrm{Str}\left(\frac{\partial U}{\partial u}V\right) = \mathrm{Str}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a & b & d \\ c & -a & e \\ e & -d & 0 \end{pmatrix} = 2a$$

$$\mathrm{Str}\left(\frac{\partial U}{\partial \epsilon}V\right) = \mathrm{Str}\begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & -\lambda & 0 \end{pmatrix}\begin{pmatrix} a & b & d \\ c & -a & e \\ e & -d & 0 \end{pmatrix} = 2\lambda e.$$

In this case, (21) becomes

$$\begin{pmatrix} \delta/\delta u \\ \delta/\delta\epsilon \end{pmatrix}(2\lambda b + 2\lambda c - 2e\epsilon) = \left(\lambda^{-\gamma}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\gamma}\right)\begin{pmatrix} 2a \\ 2\lambda e \end{pmatrix} \tag{22}$$

or

$$\begin{pmatrix} \delta/\delta u \\ \delta/\delta\epsilon \end{pmatrix}(b_n + c_n - e_n\epsilon) = (\gamma - 4)\begin{pmatrix} a_{n-1} \\ e_n \end{pmatrix}. \tag{23}$$

In particular, taking $n = 1$, we have

$$\begin{pmatrix} -2ku \\ -2k\epsilon_x \end{pmatrix} = (\gamma - 4)\begin{pmatrix} ku \\ k\epsilon_x \end{pmatrix}.$$

Therefore, $\gamma = 2$. When $n = 2$, we know from (23) that

$$\begin{pmatrix} a_1 \\ e_2 \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} \delta/\delta u \\ \delta/\delta\epsilon \end{pmatrix}(b_2 + c_2 - e_2\epsilon).$$

In general, we can write (10) in the Hamiltonian form

$$\begin{pmatrix} u \\ \epsilon \end{pmatrix}_t = \begin{pmatrix} \partial & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_n \\ \mathrm{e}_{n+1} \end{pmatrix} = \begin{pmatrix} \partial & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta/\delta u \\ \delta/\delta\epsilon \end{pmatrix} H_{n+1}$$

where $H_n = (b_n + c_n - e_n\epsilon)/(2 - 4n)$. In particular, (9) can be written in the Hamiltonian form

$$\begin{pmatrix} u \\ \epsilon \end{pmatrix}_t = \begin{pmatrix} \partial & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta/\delta u \\ \delta/\delta\epsilon \end{pmatrix} k(-\tfrac{1}{2}uu_{xx} + \tfrac{1}{4}u_x^2 + \tfrac{3}{2}u\epsilon\epsilon_{xx} - 6u_x\epsilon\epsilon_x$$
$$+\tfrac{3}{4}u^4 - \epsilon\epsilon_{xxx} + 2\epsilon_x\epsilon_{xx} + \tfrac{9}{2}u^2\epsilon\epsilon_x).$$

Remark. The supertrace identity (21) was first presented in [23]. It is noted that in [8] the supertrace identity was also applied to establish the Hamiltonian structure of superintegrable systems.

6. Concluding remarks

In this paper, an algorithm to generate integrable systems is extended to the super case. Some new examples of superextensions of integrable systems are illustrated. We also generalize the trace identity due to Tu to the super case and use it to establish Hamiltonian structures of superextensions of integrable systems under consideration. To our knowledge, the equations obtained in sections 2 and 3 are all new. It is noticed that in [7] Inami and Kanno extended the Drinfeld–Sokolov method to the supersymmetric case. As we mentioned in the introduction, there are two kinds of superextensions of the KdV equation: the so-called supersymmetric KdV equation derived by Manin and Radul and the Kupershmidt's version. The new equations found in this paper belong to the class of the Kupershmidt's superextension while the equations derived by Inami and Kanno may be viewed as to be in the class of Manin–Radul's superextension. However, as we have seen, in both cases Lie superalgebras play a key role in deriving superintegrable systems. In this paper we mainly focus on generating superextensions of integrable systems. Naturally, the algebraic and geometric properties of these new equations could be further considered. Also the corresponding recursion operators of these equations could be derived.

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